

Matrix Models: Fermion Doubling vs. Anomaly[★]

Corneliu Sochichiu^{a,b,1}

^a*Institutul de Fizică Aplicată AŞ, str. Academiei, nr. 5, Chişinău, MD2028
MOLDOVA*

^b*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear
Research, 141980 Dubna, Moscow Reg., RUSSIA²*

Abstract

We present some arguments showing spectrum doubling of matrix models in the limit $N \rightarrow \infty$ which is connected with fermionic determinant behaviour. The problems are similar to ones encountered in the lattice gauge theories with chiral fermions. One may discuss the “physical meaning” of the doubling states or ways to eliminate them. We briefly consider both situations.

Key words: Matrix model, anomaly, fermion doubling

1 Introduction

A tool for nonperturbative study of superstring theory (M-theory) proved to be matrix models, in special BFSS (M(atrix) theory) and IKKT (IIB matrix theory) ones, [1,2]. These models are formulated in terms of $N \times N$ Hermitian matrices, where $N \rightarrow \infty$.

For a finite (but large) N the last model, [1] plays the rôle of regularised “second quantised” IIB superstring in Schild formulation [3], while the former one [2] is a regularisation of $D = 11$ quantum super-membrane.

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¹ E-mail: sochichi@thsun1.jinr.ru

² Present address.

Even for finite N these models have problems with definiteness due to divergence of the partition function arising from integration along flat directions or vacua. Due to this also the limit $N \rightarrow \infty$ is badly defined. However, if not to try to consider the matrix model as a whole, but to consider a “perturbative sector” connected with fluctuations around a particular vacuum solution one may have a well defined system.

The models obtained this way depend on the chosen vacuum solution, in particular, various compactifications of IIB matrix model on noncommutative tori were shown to yield super Yang–Mills (sYM) models on these tori, as well as BFSS model, [4].

One can show that in fact Yang–Mills type models describe all regular fluctuations around a vacuum solution of IKKT matrix model, [5]. Under regular fluctuations we understand those which have bounded value of momentum and position operators. In practice e.g. numerical computations this restriction can be implemented by additional regularisation which makes the undesirable modes decouple.

Once there is a close correspondence between matrix models and the sYM models there appears the following problem. Ten dimensional sYM model is known to be anomalous and the anomaly seems to exist also in the noncommutative case, [6,7]. From the other hand there is no visible source for anomaly in the matrix model. For any finite N , the matrix model is finite manifestly gauge and Lorentz invariant as well as supersymmetric. Naively, these properties must hold also in the limit $N \rightarrow \infty$, contradicting the anomaly of $D = 10$ sYM.

The explanation of this discrepancy may reside just in the contribution of the singular configurations. In the actual paper we show that the matrix model fails to reproduce the anomaly in the limit $N \rightarrow \infty$ due to the spectrum doubling in the fermionic sector of the matrix model.

In fact the actual situation is not new. A class of models is known to suffer from the spectrum doubling, [8,9]. In particular, for lattice gauge theory, it is known that the naive discretisation of chiral fermionic action leads to doubling of the fermionic spectrum. The last results in appearance for each mode another mode(s) carrying the opposite chirality which compensate the parity odd contribution of fermions, [10]. In fact, there is a no-go theorem due to Nielsen and Ninomiya [11,12], which states that the doubling cannot be avoided unless the gauge, Lorentz or other relevant symmetry is destroyed in the continuum limit. While the gauge symmetry plays the crucial rôle in the consistency of the model, breaking of the Lorentz symmetry is not dangerous for the model and maybe even desirable as a mechanism of spontaneous breaking of the ten dimensional Lorentz group to a lower dimensional one, [13].

The plan of the paper is as follows. First we introduce the description of the fermionic sector of IIB matrix model for finite N , after that we explicitly find the doubling states for free fermionic fluctuations (quadratic approximation), and discuss the issue for the interacting case as well as possibility to eliminate the doubling.

2 Finite N Matrix Model

The IIB matrix model is described by the action,

$$S = -\frac{1}{g^2} \text{tr} \left(\frac{1}{4} [A_\mu, A_\nu]^2 + \frac{1}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \psi \right), \quad (1)$$

where A_μ and ψ are ten dimensional vector and Majorana–Weyl spinor Hermitian $N \times N$ matrices.

An important class of configurations which tend to satisfy equations of motion in the limit $N \rightarrow \infty$, are BPS ones given by $A_\mu = p_\mu$, where Hermitian matrices $p_\mu = p_\mu^\dagger$ satisfy,

$$[p_\mu, p_\nu] = iB_{\mu\nu}, \quad (2)$$

here $B_{\mu\nu}$ is proportional to the unity matrix $B_{\mu\nu} \equiv B_{\mu\nu} \cdot \mathbb{I}$.

Although, such a set of matrices does not exist for finite N , it can be approximated by a sequence of matrices converging to (2) in the sense of operator norm on the Hilbert space of smooth finite functions (vectors), [5].

Before searching such an approximation, consider a Lorentz transformation which brings matrix $B_{\mu\nu}$ to the canonical form, having 2×2 antisymmetric diagonal blocks with $\pm \hbar_i$ entries. In this case the set of matrices p_μ is split in pairs (p_i, q^i) , $i = 1, \dots, D/2$, where p_i and q^i are canonically conjugate,

$$[p_i, p_j] = [q^i, q^j] = 0, \quad (3)$$

$$[p_i, q^j] = -i\hbar_i \delta_i^j. \quad (4)$$

Eqs. (3,4) give the $D/2$ dimensional Heisenberg algebra. It is known that the Heisenberg algebra can be represented e.g. in terms of square integrable functions defined on the spectrum of operators q^i .

Finite N analog of the Heisenberg algebra (4) is given by the position and the Hermitian (symmetric) lattice derivative operator on a compact rectangular periodic lattice Γ . Let $|n\rangle$ be eigenvector of q^i with eigenvalues $L_i \sin(2\pi/N_i)$,

$$\prod_i N_i = N,$$

$$q^i |n\rangle = L_i \sin(2\pi n^i / N_i) |n\rangle, \quad (5)$$

then operators p_i act on this basis as follows,

$$p_i |n\rangle = \frac{i\hbar_i}{2a_i} (|n + e_i\rangle - |n - e_i\rangle), \quad (6)$$

where e_i is the unity lattice vector along i -th link, and $a_i = \frac{2\pi L_i}{N_i}$. The choice of q which we use differs from one usually used in lattice model by redefinition of $q \rightarrow L \sin 2\pi q / L$. It is as good as the former one, but beyond this it treats p and q in a symmetric manner. It is not difficult to see that in the basis of p eigenvectors eqs. (5) and (6) keep the same form with p and q interchanged. It also introduces explicitly the periodic boundary conditions, i.e. the identification $|n + N_i e_i\rangle \sim |n\rangle$. Due to the symmetry between p and q the continuum limit is achieved after the ‘‘UV cutoff’’ removing: $a_i \rightarrow 0$, combined with the ‘‘IR cutoff’’ removing or ‘‘decompactification’’: $L_i \rightarrow \infty$.

The commutator of p_i and q^j looks as follows,

$$\begin{aligned} [p_i, q^j] |n\rangle &= -i\hbar \delta_i^j \times \\ &\quad \frac{N}{4\pi} \left(\left\{ \sin \frac{2\pi n^j}{N_j} - \sin \frac{2\pi(n^j + 1)}{N_j} \right\} |n + e_i\rangle \right. \\ &\quad \left. + \left\{ \sin \frac{2\pi(n^j - 1)}{N_j} \right\} |n - e_i\rangle \right) \\ &\equiv -i\delta_i^j \hbar_j \mathbb{I}_{(j)}(N) |n\rangle, \end{aligned} \quad (7)$$

where we have introduced the notation $\mathbb{I}_{(j)}(N)$ for the following matrix,

$$\begin{aligned} \mathbb{I}_{(j)}(N) |n\rangle &= -\frac{N_j}{4\pi} \left(\left\{ \sin \frac{2\pi n^j}{N_j} - \sin \frac{2\pi(n^j + 1)}{N_j} \right\} |n + e_i\rangle \right. \\ &\quad \left. + \left\{ \sin \frac{2\pi(n^j - 1)}{N_j} \right\} |n - e_i\rangle \right). \end{aligned} \quad (8)$$

As it is not difficult to see the equations of motion are not satisfied by such background. It still can be shown that the $N = \infty$ solution (2) *can not* be approximated by *solutions* to equations of motion at a *finite* value of N , because at finite N the only solutions are those with zero commutator, $[A_\mu, A_\nu] = 0$.

For (sequences of) vectors $|f\rangle = \sum_n f_n |n\rangle$, on which p and q remain bounded as N goes to infinity,

$$\langle f | (p^2 + q^2) | f \rangle \leq C, \quad (9)$$

where C does not depend neither on N nor on L (or a), operator $\mathbb{I}_{(i)}(N)$ approaches the unity one,

$$\mathbb{I}_{(j)}(N)|f\rangle \approx \sum_n \frac{1}{2} \left(f_{n+e_i} \cos \frac{2\pi(n^j - 1)}{N_j} + f_{n-e_i} \cos \frac{2\pi(n^j + 1)}{N_j} \right) |n\rangle \rightarrow |f\rangle, \quad (10)$$

since in this case $n^j \ll N_j$, and $f_{n\pm e_i} - f_n = O(N^{-1})$.

The last equation means that the operators preserving the property (9) tend to commute with $\mathbb{I}_{(j)}(N)$ as N approaches the infinity.

3 Fermionic Contribution

We are ready now to proceed to the analysis of the fermionic contribution to the partition function of the model with action (1). Integration over fermionic matrices results in the Pfaffian of the fermionic operator,

$$Z(A) = \int d\psi e^{-S_f}, \quad (11)$$

where, S_f stands for the fermionic part of the action (1).

For finite N consider the bosonic background given by matrices p_i, q^i from eqs. (5,6). An arbitrary Hermitian matrix fluctuation around the given background is $A_\mu = p_\mu + ga_\mu$. The fermionic part S_f of the action in this case looks as follows,

$$S_f = -\frac{1}{2} \text{tr} \bar{\psi} \Gamma^\mu [(p_\mu + ga_\mu), \psi], \quad (12)$$

where we rescaled $\psi \rightarrow g\psi$.

The free ($a_\mu = 0$) part of the fermionic action can be written in the representation of $|n\rangle$. It looks as follows,

$$S_f = \sum_{m,n,i} \left((-i\hbar_i/(2a)) \bar{\psi}_{n,m} \Gamma^i (\psi_{m+e_i,n} - \psi_{m-e_i,n} - \psi_{m,n+e_i} + \psi_{m,n-e_i}) + L_i \bar{\psi}_{n,m} \bar{\Gamma}_i \left(\sin \frac{2\pi m^i}{N_i} - \sin \frac{2\pi n^i}{N_i} \right) \psi_{m,n} \right), \quad (13)$$

where,

$$\psi_{n,m} = \langle n | \psi | m \rangle, \quad (14)$$

$$\bar{\psi}_{m,n} = \langle m | \bar{\psi} | n \rangle, \quad (15)$$

and $\Gamma^i, \bar{\Gamma}_i$ are Dirac matrices,

$$\Gamma^i p_i + \bar{\Gamma}_i q^i \equiv \Gamma^\mu p_\mu. \quad (16)$$

Although, the action (13) differs from the naive lattice fermionic action, it shares many common features with it.

As for naive lattice fermions, they are described by the action [10],

$$S_{\text{naive}} = \frac{i}{2} \sum_n \bar{\psi}_n \Gamma^\mu (\psi_{n+e_i} - \psi_{n-e_i}), \quad (17)$$

where for shortening notations we put lattice spacing a to unity.

A well-known fact is that the actual model suffers from the fermionic spectrum doubling. The last manifests in the fact that for each chiral fermionic state there is always another one present in the spectrum with the opposite chirality but with other quantum numbers coinciding with the original state. This phenomenon can be described by introducing some discrete symmetry which relates these states. The states obtained by action of this symmetry are called doublers. It is clear that if such a symmetry exists it completely destroys the chiral asymmetry.³

In the case of action (17), such a symmetry indeed exists and its generators in a even dimension D look as follows [14],

$$T_\alpha = i\Gamma_{(D+1)}\Gamma_\alpha(-1)^{n_\alpha}, \quad (18)$$

where $\Gamma_{(D+1)}$ is the D -dimensional analog of the Dirac γ_5 -matrix (D is even), $\Gamma_{(D+1)} = \epsilon_D \Gamma^1 \Gamma^2 \dots \Gamma^D$, ϵ is chosen to be either i or 1 in order to make $\Gamma_{(D+1)}$ Hermitian.

Finding the order of the discrete group generated by T_α , one finds that the number of doubling states is $2^D - 1$.

Now, let us return back to the matrix model given by eq. (13) and try to find a similar symmetry in this case.

One can check that the action (13) is left invariant by the following symmetry,

$$\psi_{m,n} \rightarrow i\Gamma_{11}\Gamma_i(-1)^{n^i-m^i}\psi_{m,n}. \quad (19)$$

or in the matrix form,

$$\psi \rightarrow T_i \cdot \psi = i\Gamma_{11}\Gamma_i U_i^{-1} \psi U_i, \quad (19')$$

³ The absence of an explicit symmetry of such kind, however, does not prove necessarily, the absence of doubling.

where unitary matrix U_i is given by $U_i = (-1)^{\frac{N_i}{2\pi} \arcsin(q^i/L_i)}$.

Indeed, factor $\Gamma_{11}\Gamma_i$ commute with all $\bar{\Gamma}_i$ and Γ^j for $j \neq i$ while factors $(-1)^{n^i-m^i}$ are the same for both ψ and $\bar{\psi}$, and, therefore cancel. In the remaining term $\Gamma_{11}\Gamma_i$ anticommutes with Γ_i , but the extra minus sign is compensated by the variation of the factor $(-1)^{n^i-m^i}$. Thus, all terms in the action (13) remains invariant under the transformation (19).

Interchange $p \leftrightarrow q$ gives the remaining symmetries,

$$\psi \rightarrow \bar{T}_i \cdot \psi = i\Gamma_{11}\bar{\Gamma}_i\bar{U}_i^{-1}\psi\bar{U}_i, \quad (20)$$

where \bar{U}_i is the unitary transformation, $\bar{U}_i = (-1)^{\frac{N_i}{2\pi} \arcsin(pa_i/\hbar_i)}$.

Summarising one has the action (1) invariant with respect to discrete symmetry generated by T_μ ,

$$T_\mu \cdot \psi = i\Gamma_{11}\Gamma_\mu U_\mu^{-1}\psi U_\mu, \quad (21)$$

where U_μ satisfy,

$$U_\mu^{-1}p_\nu U_\mu = (1 - 2\delta_{\mu\nu})p_\nu. \quad (22)$$

As in the case with naive lattice fermions these transformations act in such a way that in the continuum limit the states become 2^D -fold degenerate with half of that for each chirality. In particular eq. (20) means that for each matrix state of given chirality which connects m and n there are $2^{D/2}$ states of different chiralities connecting $\frac{N_i}{2}e_i - m$ with $\frac{N_i}{2}e_i - n$, $\frac{N_i}{2}e_i + \frac{N_j}{2}e_j + m$ with $\frac{N_i}{2}e_i + \frac{N_j}{2}e_j + n$, $i \neq j$, and so on. Eq. (19) have the same interpretation in the “momentum space” spanned by the eigenvectors of p .

It is clear now that if one wants to compute the gauge anomaly, one will have contributions from doublers of both chiralities which cancel each other.

So far, we considered the interaction free part of the fermionic action. Presence of interaction at least in the framework of perturbation theory does not change the situation, in this case one has in the continuum limit an interacting 2^D -plet instead of a free one. Let us note, that this analysis may not remain true beyond the perturbation theory, as for strong field a_μ the interaction part dominates and the symmetry (21) of the free part does not play in this case such an important role. Unlike the usual lattice models the study of doubling in nonperturbative regime is too complicate, but the perturbative considerations are enough to doubt the result of naive continuum limit.

One may ask, what happens in this case to the supersymmetry of the original matrix model given by the action (1)?

In fact, in spite of the fermionic doubling, the supersymmetry still exists as it is valid for any matrix configuration and is irrelevant to the chosen representation. The apparent paradox with doubled number of fermions is solved if one see that the number of bosonic “degrees of freedom” is also doubled. The bosonic doublers are gauge equivalent configurations and are related by gauge transformations $a_\mu \rightarrow U_\nu^{-1} a_\mu U_\nu$.

4 Doubling States Removing

The results of the previous section show that matrix model (1) fails to reproduce a chiral continuum model in the limit $N \rightarrow \infty$. Thus, in particular, one can not obtain from it the noncommutative SYM model in this limit.

Such a situation can be interpreted either as a presence of finite N artefact which does not decouple in the limit $N \rightarrow \infty$, and must be removed by additional effort like in traditional lattice models, or as an indication that the model possesses nontrivial symmetries. In traditional lattice models the doubling states should be removed since the doubling contradict the continuum “phenomenology”, in special the chirality properties of the model.

In the case of matrix models which pretend to describe M-theory the situation is different. As it is known the M-theory unites perturbative models with different field content. In particular IIA models contain states with both chiralities while in IIB models there are only ones with definite chirality. The duality symmetry which must relate these models should contain a mechanism which flips the chirality of states. As we see, such a mechanism exists on compact noncommutative spaces and is given by doubling.

In spite of this perspective for doubling states to describe the physical reality there exists, however, possibility to remove them in order to get a chiral model in the continuum limit. Consider briefly the ways one can do this. As the problem is a “lattice” one, we can look for specific lattice solutions. In the lattice case the doubling is cured by addition of a Wilson term to the naive lattice action [10]. The problem is that there is no gauge invariant Wilson term which could be added to our “naive” fermionic action (12), as there is no gauge and Lorentz invariant fermionic mass term in the model.

One can, however, write down Wilson terms preserving either of two symmetries.

A possible gauge non-invariant Wilsonian prescription is given by addition to

the naive action (12) of the following term,

$$\Delta S_{\text{W,gauge}} = -\frac{1}{2} \text{tr} \bar{\eta} \Gamma^\mu [p_\mu, \eta] + [p_\mu^{(+)}, \bar{\eta}] [p_\mu^{(-)} \psi] + [p_\mu^{(+)}, \bar{\psi}] [p_\mu^{(-)} \eta], \quad (23)$$

where η is $U(N)$ -singlet Majorana-Weyl spinor matrix, and $p_\mu^{(\pm)}$ are respectively forward and backward one-step scaled lattice derivatives,

$$p_i^{(\pm)} |n\rangle = \pm i \sqrt{\frac{\hbar_i}{a_i}} (|n \pm e_i\rangle - |n\rangle) \quad (24)$$

$$q^{(\pm)i} |n\rangle = i \sqrt{L_i} e^{\pm 2\pi i n^i / N_i} |n\rangle. \quad (25)$$

Due to the term (23) the states with large phase of q and p , (i.e. with $n^i \sim N_i$ and $k_i \sim \frac{\hbar_i}{a_i}$) acquire large masses and decouple in the limit $N \rightarrow \infty$, as it happens in the case with usual Wilson term.

Another possibility is given by that in contrast to low-dimensional field theory models where the Lorentz invariance is obligatory, in the Matrix model it is less important. Moreover, its breaking to lower dimensional symmetries is desirable if one wants to describe a four dimensional theory in low energy limit, as it was proposed in the Ref. [13]. It is not difficult to construct Wilson term which breaks, say Lorentz group $\text{SO}(10)$ down to $\text{SO}(9)$, but preserves the gauge symmetry. It looks as follows,

$$\Delta S_{\text{W,Lorentz}} = -\frac{1}{2} \text{tr} [p_\mu^{(+)}, \bar{\psi}] \Gamma^9 [p_\mu^{(-)}, \psi], \quad (26)$$

where Γ^9 is the 9-th ten dimensional Dirac gamma matrix.

Under this choice modes with large n and k also acquires large masses in the limit $N \rightarrow \infty$, but it produces terms which are not invariant with respect to rotations involving the 9-th axis. The gauge symmetry here remains intact. The last should not appear strange, because giving up a part of Lorentz invariance in an anomalous model allows one to cancel anomaly. Thus, in the simplest case of Abelian gauge anomaly in $D = 4$ one can cancel the anomaly $\sim \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}$, where $\mu, \nu, \dots = 0, 1, 2, 3$ by addition a local counterterm $\sim A_0 \epsilon^{ijk} A_i \partial_j A_k$, with $i, j, \dots = 1, 2, 3$ which is not invariant with respect to Lorentz boosts.

5 Discussions

We have shown that perturbative fluctuations in IIB matrix model at finite N exhibit a phenomenon similar to one in lattice gauge theories with fermions, consisting in doubling of the fermionic spectrum.

We considered a simple example of a background given by lattice shift/momentum shift operators. Basing on analogy with lattice model, we conjecture that this is a universal feature appearing for arbitrary choice of background configuration p_μ which is Hermitian, nondegenerate, etc., and can not be eliminated without breaking either Lorentz or gauge invariance. The actual results are straightforwardly translated to the case of the genuine finite N vacua considered in [5].

We give prescriptions for the elimination of the doubling states in the limit $N \rightarrow \infty$, but preserving either Lorentz or gauge invariance and, respectively, breaking another one. This prescriptions should break the supersymmetry, since they do not restrict the bosonic spectrum as well.

However, in contrast to lattice gauge models, one may *not need* to eliminate such doubling states. Since there is a conjecture that IKKT matrix model nonperturbatively describes both IIB and IIA string models [1], which have different chirality content, it may be possible that the doubling in the language of matrix models is related to the string duality.

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